411 **a.** Calcola, dopo averne opportunamente semplificato l'espressione, le soluzioni z_1, z_2, z_3 dell'equazione $(1+i)z^3 = 8\sqrt{2}\,i$, con $z \in \mathbb{C}$.

b. Calcola $z_1^2 + z_2^2 + z_3^2$.

[a) $z_1 = 2(\cos 15° + i \cdot \sin 15°)$, $z_2 = 2(\cos 135° + i \cdot \sin 135°)$, $z_3 = 2(\cos 255° + i \cdot \sin 255°)$; b) 0]

a) $(1+i)z^3 = 8\sqrt{2}\,i$

$$z^3 = \frac{8\sqrt{2}\,i}{1+i} \cdot \frac{1-i}{1-i} = \frac{8\sqrt{2}\,i + 8\sqrt{2}}{1+1} = \frac{8\sqrt{2} + 8\sqrt{2}\,i}{2} = 4\sqrt{2} + 4\sqrt{2}\,i$$

Calcolo le radici cubiche di $4\sqrt{2} + 4\sqrt{2}\,i$

$$\rho = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{32+32} = \sqrt{64} = 8$$

$$4\sqrt{2} + 4\sqrt{2}\,i = 8\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right) = 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$z_0 = \sqrt[3]{8}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) = 2\left(\frac{\sqrt{6}+\sqrt{2}}{4} + i\,\frac{\sqrt{6}-\sqrt{2}}{4}\right) = \frac{\sqrt{6}+\sqrt{2}}{2} + i\,\frac{\sqrt{6}-\sqrt{2}}{2}$$

$$z_1 = \sqrt[3]{8}\left[\cos\left(\frac{\pi}{12} + \frac{2}{3}\pi\right) + i\sin\left(\frac{\pi}{12} + \frac{2}{3}\pi\right)\right] =$$

$$= 2\left[\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right] = 2\left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = -\sqrt{2} + i\sqrt{2}$$

$$z_2 = \sqrt[3]{8}\left[\cos\left(\frac{\pi}{12} + \frac{4}{3}\pi\right) + i\sin\left(\frac{\pi}{12} + \frac{4}{3}\pi\right)\right] =$$

$$= 2\left[\cos\frac{17}{12}\pi + i\sin\frac{17}{12}\pi\right] = 2\left[\cos\left(\pi + \frac{5}{12}\pi\right) + i\sin\left(\pi + \frac{5}{12}\pi\right)\right] =$$

$$= 2\left[-\cos\frac{5}{12}\pi - i\sin\frac{5}{12}\pi\right] = 2\left[-\cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{2} - \frac{\pi}{12}\right)\right] =$$

$$= 2\left[-\sin\frac{\pi}{12} - i\cos\frac{\pi}{12}\right] = 2\left[-\frac{\sqrt{6}-\sqrt{2}}{4} - i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right] =$$

$$= \frac{\sqrt{2}-\sqrt{6}}{2} - i\,\frac{\sqrt{6}+\sqrt{2}}{2}$$

b) $z_0^2 + z_1^2 + z_2^2 =$

$$= \left(\frac{\sqrt{6}+\sqrt{2}}{2} + i\,\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 + \left(-\sqrt{2} + i\sqrt{2}\right)^2 + \left(\frac{\sqrt{2}-\sqrt{6}}{2} - i\,\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 =$$

$$= 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)^2 + 4\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)^2 + 4\left(\cos\frac{17}{12}\pi + i\sin\frac{17}{12}\pi\right)^2 =$$

$$= 4\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} + \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi + \cos\frac{17}{6}\pi + i\sin\frac{17}{6}\pi\right] =$$

$$= 4\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i - i + \cos\left(3\pi - \frac{\pi}{6}\right) + i\sin\left(3\pi - \frac{\pi}{6}\right)\right] =$$

$$= 4\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i + \cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right] =$$

$$= 4\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i - \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] =$$

$$= 4\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right] = \boxed{0}$$

413 È data l'equazione $z^5 - z^4 + 9z - 9 = 0$, dove $z \in \mathbb{C}$. Verifica che $z = 1$ è una radice e dopo avere abbassato il grado dell'equazione determina le restanti radici. Rappresenta le soluzioni nel piano di Gauss.

$$\left[\frac{\sqrt{6}}{2}(1+i), \frac{\sqrt{6}}{2}(-1+i), \frac{\sqrt{6}}{2}(-1-i), \frac{\sqrt{6}}{2}(1-i)\right]$$

$$z^5 - z^4 + 9z - 9 = 0$$

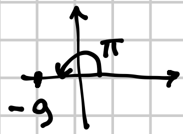
$$z^4(z-1) + 9(z-1) = 0$$

1 è soluzione

$$(z-1)(z^4+9) = 0 \qquad z = 1 \quad \vee \quad z^4 = -9$$

devo trovare le 4 radici 4° di -9

$$W = -9 = 9 \cdot (-1) = 9(\cos\pi + i\sin\pi)$$



$$W_0 = \sqrt[4]{9}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{3}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{2} + i\frac{\sqrt{6}}{2}$$

$$W_1 = \sqrt{3}\left(\cos\left(\frac{\pi}{4} + \underbrace{\frac{\pi}{2}}_{\frac{2\pi}{4}}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right) = \sqrt{3}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right) =$$

$$= \sqrt{3}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2} + i\frac{\sqrt{6}}{2}$$

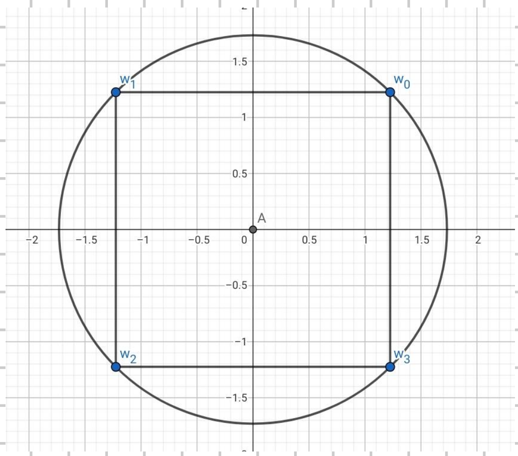
$$W_2 = \sqrt{3}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right) = \sqrt{3}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{2} - i\frac{\sqrt{6}}{2}$$

$$W_3 = \sqrt{3}\left(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi\right) = \sqrt{3}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{2} - i\frac{\sqrt{6}}{2}$$

$$\boxed{z = 1 \quad \vee \quad z = \frac{\sqrt{6}}{2} \pm i\frac{\sqrt{6}}{2} \quad \vee \quad z = -\frac{\sqrt{6}}{2} \pm i\frac{\sqrt{6}}{2}}$$

REALE

A 2 A 2 CONIUGATE



414 Dato $z \in \mathbb{C}$, sia $\overline{z}$ il suo complesso coniugato. Rappresenta nel piano di Gauss l'insieme $E \cap F$, con:

$$E = \{z \in \mathbb{C}: |z-1| < |\overline{z}|\}, \quad F = \left\{z \in \mathbb{C}: \left|z - \frac{1}{2}\right| \le 2\right\}.$$

$$E = \{z \in \mathbb{C} : |z-1| < |\overline{z}|\} \qquad z = x + iy$$

$$|x+iy-1| < |x-iy|$$

$$|(x-1)+iy| < |x-iy|$$

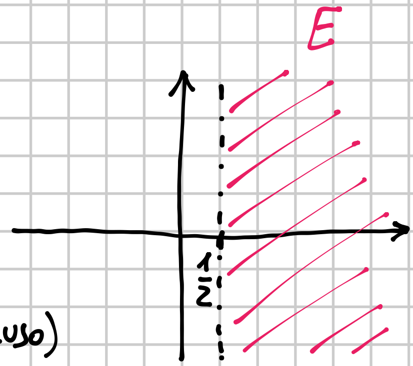
$$\sqrt{(x-1)^2+y^2} < \sqrt{x^2+y^2}$$

$$(x-1)^2+\cancel{y^2} < x^2+\cancel{y^2}$$

$$\cancel{x^2}+1-2x < \cancel{x^2}$$

$$1-2x<0 \Rightarrow 2x>1 \Rightarrow x > \frac{1}{2}$$

SEMIPIANO (BORDO ESCLUSO)

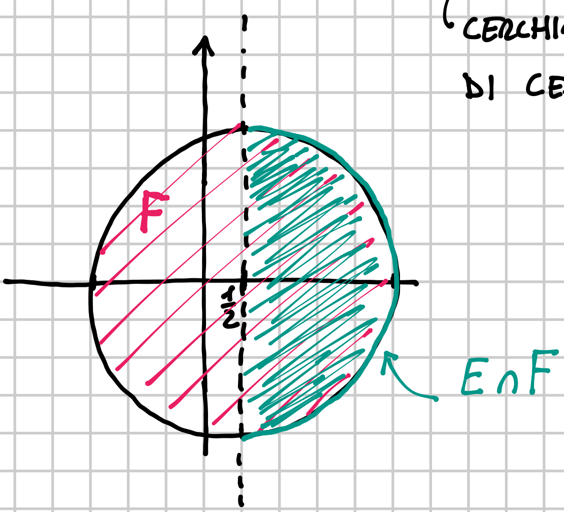


$$F = \{z \in \mathbb{C} \mid |z - \tfrac{1}{2}| \le 2\} \qquad z = x+iy$$

$$\left|x+iy-\frac{1}{2}\right| \le 2 \qquad \left|\left(x-\frac{1}{2}\right)+iy\right| \le 2$$

$$\sqrt{\left(x-\frac{1}{2}\right)^2+y^2} \le 2 \qquad \left(x-\frac{1}{2}\right)^2+y^2 \le 4$$

CERCHIO (PARTE INTERNA) DI CENTRO $\left(\frac{1}{2}, 0\right)$ E RAGGIO 2



410 $z_1 = \sqrt{2} + i, \qquad z_2 = \sqrt{2} - i.$

Scrivere un'equazione di 2° grado che abbia z_1, z_2 come soluzioni

$$(z - z_1)(z - z_2) = 0$$

$$z^2 - z_2 z - z_1 z + z_1 z_2 = 0$$

$$z^2 - (z_1 + z_2) z + z_1 z_2 = 0$$

$$z^2 - (\sqrt{2} + \cancel{i} + \sqrt{2} - \cancel{i}) z + (2 - i^2) = 0$$

$$\boxed{z^2 - 2\sqrt{2}\, z + 3 = 0}$$

$$\frac{\Delta}{4} = 2 - 3 = -1$$

le 2 radici quadrate di -1 sono $\pm i$

$$z = \sqrt{2} \pm i$$